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On some characterizations of inner product spaces ☆

José Mendoza * and Tijani Pakhrou

Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040 Madrid, Spain

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Abstract

We study some characterizations of inner product spaces given in the literature. Among other things, we give an example showing that one of the characterizations given in the classical book of Amir (1986) is not correct.

0. Introduction

This paper has been motivated by the study of [3].

In Section 4 of [3], Durier extends a couple of characterizations of inner product spaces (in short, IPS) given by Amir in his classical book [1]. Our study led us to try to understand as well as possible Amir's original characterizations, and so let us begin with them. We first need some notations.

Let $(X, \|\cdot\|)$ be a real normed linear space, let x_0 be an element of X and let A be a nonempty bounded subset of X. We denote

$$r(x_0, A) = \sup\{\|y - x_0\|:\ y \in A\}$$

and

$$r(A) = \inf\{r(x, A):\ x \in X\}.$$

The number $r(A)$ is called the Chebyshev radius of A, and we denote

$$Z(A) = \{x \in X:\ r(x, A) = r(A)\}.$$

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* Corresponding author.

E-mail addresses: jose_mendoza@mat.ucm.es (J. Mendoza), tijani_pakhrou@mat.ucm.es (T. Pakhrou).

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This set (possibly empty) is the Chebyshev center set of A. Any point of $Z(A)$ is said a Chebyshev center of A. Given $r > 0$ we denote by $B(x_0, r)$ the closed ball centered at x_0 with radius r, that is,

$$B(x_0, r) = \left\{x \in X:\ \|x - x_0\| \leqslant r\right\}.$$

Notice that if x_0 is a Chebyshev center of A and we take $r = r(A)$, then

$$A \subset B(x_0, r).$$

In fact, the Chebyshev radius of A, $r(A)$, is the smallest number $r \geqslant 0$ for which there exists $x \in X$ such that

$$A \subset B(x, r).$$

We can now state Amir's announced characterizations. They are motivated by Garkavi–Klee theorem (see, for instance, (15.1) and (15.2) of [1]). In particular, the first one is intended as a sharpening of such result. We keep the numbers assigned in the book.

Theorem (Amir [1]). *Let X be a normed linear space of dimension at least three, then X is an IPS if and only if any of the two following conditions holds*:

(15.14) *If a_1, a_2, a_3 are norm one points in X such that $r(\{a_1, a_2, a_3\}) = 1$, then $\mathbf{0}$ is in the convex hull of $\{a_1, a_2, a_3\}$.*

(15.15) *If a_1, a_2, a_3 are norm one points in X such that $\mathbf{0}$ is in the convex hull of $\{a_1, a_2, a_3\}$, then $r(\{a_1, a_2, a_3\}) = 1$.*

Since a_1, a_2, a_3 are norm one points, $r(\{a_1, a_2, a_3\}) = 1$ just means that $\mathbf{0}$ is Chebyshev center of $\{a_1, a_2, a_3\}$. Therefore, the following conditions are just reformulations of (15.14) and (15.15):

(15.14′) *If a_1, a_2, a_3 are norm one points in X and $\mathbf{0}$ is Chebyshev center of the set $\{a_1, a_2, a_3\}$, then $\mathbf{0}$ is in its convex hull.*

(15.15′) *If a_1, a_2, a_3 are norm one points in X such that $\mathbf{0}$ is in the convex hull of $\{a_1, a_2, a_3\}$, then $\mathbf{0}$ is Chebyshev center of it.*

When we tried to understand well characterizations (15.14) and (15.15) we found out that we were not able to reproduce Amir's proofs. Of course, if X is an IPS then conditions (15.14) and (15.15) hold. The problem is how to prove converses. Amir just say that they are "immediate by Lemma 15.1," but we did not see how.

We could prove, following the idea of Lemma 15.1, that for finite-dimensional spaces, and even for reflexive Banach spaces, condition (15.14) does imply "X is an IPS," but we did not get a proof for the general case. Finally we have concluded that there was some mistake: condition (15.14) *does not* imply "X is an IPS." We have found an example and this is the content of Section 1.

Concerning condition (15.15), we could give a proof showing that actually it *does* imply "X is an IPS," but we believe it has nothing to do with Lemma 15.1. We have included it in Section 2.

It should be pointed out that there is a misprint in the statement of the aforementioned Lemma 15.1 of [1]: its last line should read "$r(p,\Delta) = r_{\mathrm{span}(p,\Delta)}(\Delta) < r_{\mathrm{conv}\Delta}(\Delta)$," instead of "$r(z,\Delta) = r_{\mathrm{span}(z,\Delta)}(\Delta) < r_{\mathrm{conv}\Delta}(\Delta)$." This is transparent from the proof of the very lemma (and from the fact that the lemma as written is meaningless).

Finally, our third and last section is more directly involved with the work of Durier in [3]. We have deepened in the study of one of the properties on optimal locations introduced by him, namely $(\mathrm{AfHP})_n^\gamma$. We show that, excluding trivial cases, it is only enjoyed by IPS.

1. An example on Amir's characterization (15.14)

To construct our example, we need some previous results.

Recall that if C is a convex subset of a vector space X the Minkowski functional (or gauge) μ_C associated to C is defined for each $x \in X$ by

$$\mu_C(x) = \inf\left\{\rho > 0\colon \frac{1}{\rho}x \in C\right\}.$$

We will exploit the close relationship between (semi)norms and the Minkowski functional of convex sets (see, for instance, 1.33, 1.34, 1.35 of [6]). In particular, we will use the following well known result (a proof is included for the sake of completeness).

Lemma 1. *Let $(X, \|\cdot\|)$ be a normed linear space and let $B_X = \{x \in X\colon \|x\| \leqslant 1\}$ its unit ball, then*

$$\mu_{B_X}(x) = \|x\|$$

for each $x \in X$.

Proof. Of course $\mu_{B_X}(\mathbf{0}) = \|\mathbf{0}\| = 0$. Let us take now $x \neq \mathbf{0}$. The equality $\|x/\|x\|\| = 1$ implies that $x/\|x\|$ belongs to B_X, and so $\mu_{B_X}(x) \leqslant \|x\|$. On the other hand, if $\frac{1}{\rho}x$ belongs to B_X this means that we have

$$\left\|\frac{1}{\rho}x\right\| = \frac{1}{\rho}\|x\| \leqslant 1$$

and, therefore, $\|x\| \leqslant \rho$. Hence $\|x\| \leqslant \mu_{B_X}(x)$. □

We can prove now our main lemma. Its geometrical meaning should be quite clear: it is a way of extending a norm "preventing" $\mathbf{0}$ to be Chebyshev center of a given triplet of norm one points. The procedure works whenever the α in the statement is smaller than 1.

Given a subset D of a linear space, we will denote by $\mathrm{conv}(D)$ the convex hull of D.

Lemma 2. *Let X be a vector space and let Y be a hyperplane in X. Let $\|\cdot\|$ be a norm in Y and assume that a_1, a_2, a_3 are three different norm one points in Y. Let u be a point in $X \setminus Y$ and let us denote*

$$\alpha = \max\left\{\left\|\frac{a_i - a_j}{2}\right\|\colon 1 \leqslant i, j \leqslant 3\right\}.$$

Then there exists a norm $\|\cdot\|_0$ *in* X *such that*:

(i) $\|y\|_0 = \|y\|$ *for each* $y \in Y$, *and*
(ii) $\|u - a_i\|_0 \leqslant \alpha$ *for each* $i = 1, 2, 3$.

Proof. Let B_Y be the closed unit ball of $(Y, \|\cdot\|)$, and let us take

$$A = \bigcup_{i=1}^{3} \left\{ -\frac{1}{\alpha}(u - a_i), \frac{1}{\alpha}(u - a_i) \right\} \quad \text{and} \quad B = \operatorname{conv}(B_Y \cup A).$$

It is clear that B is a balanced, absorbing and convex subset of X which contains no non-trivial subspaces. Therefore, its *Minkowski functional* is a norm on X. Let us denote it by $\|\cdot\|_0$. Obviously, it satisfies (ii). Hence we only have to show that it satisfies (i), too.

The main point is to show that B_Y and $B \cap Y$ coincide.

Of course, $B_Y \subseteq B \cap Y$. Let us prove $B \cap Y \subseteq B_Y$. Notice first that B can be seen as the convex hull of

$$B_Y \cup \operatorname{conv}(A^+) \cup \operatorname{conv}(A^-),$$

where

$$A^+ = \left\{ \frac{1}{\alpha}(u - a_1), \frac{1}{\alpha}(u - a_2), \frac{1}{\alpha}(u - a_3) \right\} \quad \text{and} \quad A^- = -A^+.$$

Take y in $B \cap Y$. Since y belongs to B, there exist $y_1 \in B_Y$, $y_2 \in \operatorname{conv}(A^+)$ and $y_3 \in \operatorname{conv}(A^-)$, and nonnegative numbers λ_1, λ_2, λ_3, such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3$. Now, since $y_2 \in \operatorname{conv}(A^+)$ and $y_3 \in \operatorname{conv}(A^-)$, we deduce that there exist nonnegative numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, such that $\sum_{i=1}^3 \alpha_i = \sum_{i=1}^3 \beta_i = 1$, $y_2 = \sum_{i=1}^3 \alpha_i \frac{1}{\alpha}(u - a_i)$ and $y_3 = \sum_{i=1}^3 \beta_i \frac{1}{\alpha}(a_i - u)$. Therefore, we have

$$y = \lambda_1 y_1 + \frac{\lambda_2}{\alpha} \sum_{i=1}^3 \alpha_i (u - a_i) + \frac{\lambda_3}{\alpha} \sum_{j=1}^3 \beta_j (a_j - u).$$

Hence,

$$y = \lambda_1 y_1 + \frac{1}{\alpha}(\lambda_2 - \lambda_3) u + \frac{\lambda_3}{\alpha} \sum_{j=1}^3 \beta_j a_j - \frac{\lambda_2}{\alpha} \sum_{i=1}^3 \alpha_i a_i.$$

Since y, y_1 and the a_i's do belong to Y, but u does not, it follows that $\lambda_2 - \lambda_3 = 0$. Let us denote $\mu = \lambda_2 = \lambda_3$. We have

$$y = \lambda_1 y_1 + \frac{\mu}{\alpha} \left(\sum_{j=1}^3 \beta_j a_j - \sum_{i=1}^3 \alpha_i a_i \right).$$

Therefore, $y = \lambda_1 y_1 + 2\mu y_0$, where

$$y_0 = \frac{1}{2\alpha} \left(\sum_{j=1}^3 \beta_j a_j - \sum_{i=1}^3 \alpha_i a_i \right).$$

For each $i, j \in \{1, 2, 3\}$, put $y_{ij} = \frac{1}{2\alpha}(a_j - a_i)$. It is immediate that

$$y_0 = \sum_{j=1}^{3} \beta_j \left(\sum_{i=1}^{3} \alpha_i y_{ij} \right).$$

But each y_{ij} belongs to B_Y, we deduce that y_0 belongs to B_Y, too. Now, from the equality $y = \lambda_1 y_1 + 2\mu y_0$ and the fact that $\lambda_1 + 2\mu = \lambda_1 + \lambda_2 + \lambda_3 = 1$, it follows that y belongs to B_Y, as we wished.

Finally, let us show that the equality $B \cap Y = B_Y$ implies that (i) holds. Notice that given $y \in Y$ and a nonzero real number t, it is clear that $ty \in B \cap Y$ if and only if $ty \in B$. Therefore, using the preceding lemma, for each $y \in Y$ we have

$$\begin{aligned} \|y\|_0 &= \inf\left\{ \rho > 0\colon\ \frac{1}{\rho} y \in B \right\} = \inf\left\{ \rho > 0\colon\ \frac{1}{\rho} y \in B \cap Y \right\} \\ &= \inf\left\{ \rho > 0\colon\ \frac{1}{\rho} y \in B_Y \right\} = \|y\|. \qquad \square \end{aligned}$$

Now we add strict convexity to the preceding result.

Lemma 3. *Let X be a vector space and let Y be a hyperplane in X. Let $\|\cdot\|$ be a strictly convex norm in Y and assume that a_1, a_2, a_3 are three different norm one points in Y such that $\mathbf{0} \notin \mathrm{conv}(\{a_1, a_2, a_3\})$. Let u be a point in $X \backslash Y$ and let us denote*

$$\alpha = \max\left\{ \left\| \frac{a_i - a_j}{2} \right\| \colon\ 1 \leqslant i, j \leqslant 3 \right\}.$$

Then there exists a strictly convex norm $\|\cdot\|_s$ in X such that:

(i) $\|y\|_s = \|y\|$ *for each* $y \in Y$, *and*
(ii) $\|u - a_i\|_s < (\alpha + 1)/2$ *for each* $i = 1, 2, 3$.

Proof. It follows from the preceding lemma that there exists a norm $\|\cdot\|_0$ in X verifying $\|y\|_0 = \|y\|$ for $y \in Y$ and $\|u - a_i\|_0 \leqslant \alpha$ a for $i = 1, 2, 3$.

Since Y is a hyperplane in X and u does not belong to Y we can define in X the norm

$$|||y + \theta u||| = \left(\|y\|^2 + \theta^2 \right)^{1/2}$$

for $y \in Y$ and $\theta \in \mathbb{R}$. This norm is strictly convex because $\|\cdot\|$ is strictly convex. Besides, it is obviously an extension of $\|\cdot\|$.

On the other hand, since $\mathbf{0} \notin \mathrm{conv}(\{a_1, a_2, a_3\})$, it is clear that $\mathbf{0} \neq a_i + a_j$, and so $a_i \neq -a_j$ for $i, j = 1, 2, 3$. Therefore, it follows from the strict convexity of $\|\cdot\|$ that

$$\alpha < 1.$$

Now, we can take $\epsilon > 0$ such that

$$(1 - \epsilon)\alpha + \epsilon\sqrt{2} < \frac{1 + \alpha}{2},$$

because $\lim_{\lambda \to 0^+} (1 - \lambda)\alpha + \lambda\sqrt{2} = \alpha < (1 + \alpha)/2$.

Consider in X the norm

$$\|x\|_s = (1-\epsilon)\|x\|_0 + \epsilon |||x|||.$$

This norm is strictly convex, as a consequence of the strict convexity of $|||\cdot|||$. Moreover, (i) holds because $\|\cdot\|_0$ and $|||\cdot|||$ coincide with $\|\cdot\|$ in Y.

Finally, for each $i = 1, 2, 3$ we have

$$\|u - a_i\|_s = (1-\epsilon)\|u - a_i\|_0 + \epsilon |||u - a_i||| \leqslant (1-\epsilon)\alpha + \epsilon\sqrt{2} < \frac{1+\alpha}{2};$$

hence (ii) is satisfied, too. □

Remark 1. Let us suppose we are in the hypothesis of the preceding lemma. Then, as pointed out in the proof, we have $\alpha < 1$, and therefore,

$$\frac{\alpha+1}{2} < 1.$$

One should realize that, thanks to this last inequality, (ii) in the lemma implies that $\mathbf{0}$ is not Chebyshev center of $\{a_1, a_2, a_3\}$ in $(X, \|\cdot\|_s)$.

We can give now the construction of our example.

Example. There exists a real normed linear space $(X, \|\cdot\|)$ such that

(1) $(X, \|\cdot\|)$ is *not* an IPS.
(2) $(X, \|\cdot\|)$ satisfies (15.14).

We will use the following reformulation of (15.14) (or equivalently, (15.14′)):

(15.14″) *If a_1, a_2, a_3 are three norm one points in X such that $\mathbf{0} \notin \operatorname{conv}(\{a_1, a_2, a_3\})$, then there exists $u \in X$ such that $\|u - a_i\| < 1$ for $i = 1, 2, 3$.*

Let us try first to explain the idea of our construction. Take as starting point a strictly convex three-dimensional normed linear space which is not an IPS. Remember that for finite-dimensional spaces, to be IPS is equivalent to (15.14″). So, (15.14″) does *not* hold. Therefore, we know that in our space there exists what we may call a "bad triplet": a triplet a_1, a_2, a_3 of norm one points in X such that $\mathbf{0} \notin \operatorname{conv}(\{a_1, a_2, a_3\})$, and such that there is no u in our space satisfying $\|u - a_i\| < 1$ for $i = 1, 2, 3$. But we wish (15.14″) to hold, and so we do not wish "bad triplets" to exist. So we can apply Lemma 3. It adds one dimension to our space in such a way that a_1, a_2, a_3 is no longer a "bad triplet." Of course this is not enough, because our space have not only one, but many "bad triplets." Using separability we can get something like a "dense" sequence of "bad triplets," and this will lead us to some construction by induction. We have however an additional problem: each time we add one dimension to our space the set of "bad triplets" grows. We find a construction which solves these problems simultaneously.

Let us begin with the construction of our example.

Our space X will be the vector space of all sequences of real numbers with finitely many nonzero terms. In this vector space we will consider the three-dimensional vector space X_1 given by

$$X_1 = \{(t_1, t_2, t_3, 0, 0, \dots)\colon\ t_1, t_2, t_3 \in \mathbb{R}\}$$

which is a hyperplane in

$$X_2 = \{(t_1, t_2, t_3, t_4, 0, 0, \dots)\colon\ t_1, t_2, t_3, t_4 \in \mathbb{R}\}.$$

In general, for each natural number $n \geqslant 1$ we will take

$$X_n = \{(t_1, t_2, \dots, t_n, t_{n+1}, t_{n+2}, 0, 0, \dots)\colon\ t_1, t_2, \dots, t_n, t_{n+1}, t_{n+2} \in \mathbb{R}\}.$$

Thus

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$$

and each X_n is a hyperplane in X_{n+1}. The space X is the union of this increasing sequence of finite-dimensional spaces.

We have said that we take as starting point "a strictly convex three-dimensional normed linear space which is not an IPS." So we take in X_1 a strictly convex non-Euclidean norm, which will be denoted by $\|\cdot\|_1$. Of course, this norm $\|\cdot\|_1$ has nothing to do with the ℓ_1-norm (which is not strictly convex). For instance, it may be any ℓ_p-norm with $1 < p < \infty$, $p \neq 2$.

Since the unit sphere of $(X_1, \|\cdot\|_1)$ is separable, there exists a sequence

$$\{(a^1_{(1,k)}, a^2_{(1,k)}, a^3_{(1,k)})\}_{k\geqslant 1}$$

of triplets of norm one vectors in $(X_1, \|\cdot\|_1)$ such that for each $\epsilon > 0$ and for each triplet (x^1, x^2, x^3) of norm one vectors in $(X_1, \|\cdot\|_1)$ there exists $k \in \mathbb{N}$ such that

$$\|x^i - a^i_{(1,k)}\|_1 < \epsilon$$

for $i = 1, 2, 3$.

Clearly, we can assume that for each k the vectors $a^1_{(1,k)}, a^2_{(1,k)}, a^3_{(1,k)}$ are *linearly independent*. We we will assume this. Hence $\mathbf{0} \notin \mathrm{conv}(\{a^1_{(1,k)}, a^2_{(1,k)}, a^3_{(1,k)}\})$.

For our inductive construction, we need a bijection π from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$ enjoying the following property: for each $n \in \mathbb{N}$, if we denote $\pi(n) = (m, k) \in \mathbb{N} \times \mathbb{N}$, we have $m \leqslant n$. We can take, for example, the usual enumeration of $\mathbb{N} \times \mathbb{N}$ (see, for instance, [2, Appendix A: A6]): $(1,1)$, $(1,2)$, $(2,1)$, $(3,1)$, $(2,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$, $(5,1), \dots$.

Let us apply now the preceding lemma to the space X_2. We take X_1 as the hyperplane Y. The property enjoyed by π guarantees that $\pi(1)$ has the form $(1, k)$, hence the triplet $(a^1_{\pi(1)}, a^2_{\pi(1)}, a^3_{\pi(1)})$ is in the sequence taken above. We take this triplet as the norm one points and $e_4 = (0, 0, 0, 1, 0, \dots)$ as the point u in $X \backslash Y$. It follows from the lemma that there exists a strictly convex norm $\|\cdot\|_2$ in X_2 such that:

(i) $\|x\|_1 = \|x\|_2$ for each $x \in X_1$, and
(ii) $\|e_4 - a^i_{\pi(1)}\|_2 < (\alpha_{\pi(1)} + 1)/2$ for each $i = 1, 2, 3$,

where

$$\alpha_{\pi(1)} = \max\left\{ \left\| \frac{a^i_{\pi(1)} - a^j_{\pi(1)}}{2} \right\|_1 : 1 \leqslant i, j \leqslant 3 \right\}.$$

Let us repeat the preceding procedure with $(X_2, \|\cdot\|_2)$. There exists a sequence

$$\left\{ \left(a^1_{(2,k)}, a^2_{(2,k)}, a^3_{(2,k)} \right) \right\}_{k \geqslant 1}$$

of linearly independent triplets such that for each $\epsilon > 0$ and each triplet (x^1, x^2, x^3) of norm one vectors in $(X_2, \|\cdot\|_2)$ there exists $k \in \mathbb{N}$ such that

$$\left\| x^i - a^i_{(2,k)} \right\|_2 < \epsilon$$

for $i = 1, 2, 3$. Notice once more that linear independence guarantees that $\mathbf{0} \notin \operatorname{conv}(\{a^1_{(2,k)}, a^2_{(2,k)}, a^3_{(2,k)}\})$.

By the property enjoyed by π, the pair $\pi(2)$ has either the form $(1, k)$ or the form $(2, k)$. Hence the triplet $(a^1_{\pi(2)}, a^2_{\pi(2)}, a^3_{\pi(2)})$ is in one of the two sequences taken above, and in any case the $a^i_{\pi(2)}$'s are norm one points in $(X_2, \|\cdot\|_2)$.

The preceding lemma provides us a strictly convex norm $\|\cdot\|_3$ in X_3 such that:

(i) $\|x\|_2 = \|x\|_3$ for each $x \in X_2$, and
(ii) $\|e_5 - a^i_{\pi(2)}\|_3 < (\alpha_{\pi(2)} + 1)/2$ for each $i = 1, 2, 3$,

where $e_5 = (0, 0, 0, 0, 1, 0, \ldots)$ and

$$\alpha_{\pi(2)} = \max\left\{ \left\| \frac{a^i_{\pi(2)} - a^j_{\pi(2)}}{2} \right\|_2 : 1 \leqslant i, j \leqslant 3 \right\}.$$

Then, by induction, we get in each vector space X_n a strictly convex norm $\|\cdot\|_n$ and a sequence of $\|\cdot\|_n$-norm one linearly independent triplets

$$\left\{ \left(a^1_{(n,k)}, a^2_{(n,k)}, a^3_{(n,k)} \right) \right\}_{k \geqslant 1}$$

such that for each triplet of norm one vectors (x^1, x^2, x^3) in $(X_n, \|\cdot\|_n)$ and for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$\left\| x^i - a^i_{(n,k)} \right\|_n < \epsilon$$

for $i = 1, 2, 3$. By the construction we also have:

(a) For each $n \in \mathbb{N}$, the norm $\|\cdot\|_{n+1}$ is an extension of $\|\cdot\|_n$, that is, $\|x\|_{n+1} = \|x\|_n$ for all $x \in X_n$, and
(b) For each $n \in \mathbb{N}$

$$\left\| e_{n+3} - a^i_{\pi(n)} \right\|_{n+1} < \frac{\alpha_{\pi(n)} + 1}{2}$$

for $i = 1, 2, 3$, where $e_{n+3} = (\overbrace{0, 0, \ldots, 0}^{n+2}, 1, 0, 0, \ldots)$ and

$$\alpha_{\pi(n)} = \max\left\{ \left\| \frac{a^i_{\pi(n)} - a^j_{\pi(n)}}{2} \right\|_n : 1 \leqslant i, j \leqslant 3 \right\}.$$

As we know that $\pi(n)$ has the form (m,k) with $1 \leqslant m \leqslant n$, the $a^i_{\pi(n)}$'s are norm one points in some of the spaces $(X_1, \|\cdot\|_1), \ldots, (X_n, \|\cdot\|_n)$. Moreover, since each norm is an extension of the preceding one, we can assure that they are norm one points in $(X_n, \|\cdot\|_n)$.

In $X = \bigcup_{n=1}^{\infty} X_n$, we define the norm

$$\|x\| = \|x\|_n$$

for $x \in X_n$. This norm coincides with $\|\cdot\|_n$ in each X_n. Hence, it is a well defined strictly convex norm.

Since $(X_1, \|\cdot\|_1)$ is a subspace of $(X, \|\cdot\|)$ which is not an IPS, it is clear that $(X, \|\cdot\|)$ is not an IPS either. Therefore, it satisfies (1).

It only remains to show that (2) also holds. Let us see that (15.14″) is satisfied. Let (a^1, a^2, a^3) be a triplet of norm one unit vectors in $(X, \|\cdot\|)$ such that $\mathbf{0} \notin \operatorname{conv}(\{a^1, a^2, a^3\})$. There exists $m \in \mathbb{N}$ such that (a^1, a^2, a^3) are unit vectors in $(X_m, \|\cdot\|_m)$. Since $\mathbf{0} \notin \operatorname{conv}(\{a^1, a^2, a^3\})$ and $\|\cdot\|$ is strictly convex, we have

$$\alpha = \max\left\{ \left\| \frac{a^i - a^j}{2} \right\| \colon 1 \leqslant i, j \leqslant 3 \right\} < 1.$$

Let $\epsilon > 0$ be such that $\alpha + 3\epsilon < 1$, and let $k \in \mathbb{N}$ be such that $\|a^i - a^i_{(m,k)}\|_m = \|a^i - a^i_{(m,k)}\| < \epsilon$ for $i = 1, 2, 3$. It is immediate that $\alpha_{(m,k)} < \alpha + \epsilon$. Let us take now $n \in \mathbb{N}$ such that $\pi(n) = (m,k)$. Remember that $m \leqslant n$. By (b), we have

$$\left\| e_{n+3} - a^i_{\pi(n)} \right\|_{n+1} = \left\| e_{n+3} - a^i_{(m,k)} \right\| < \frac{\alpha_{(m,k)} + 1}{2},$$

and therefore, for $i = 1, 2, 3$,

$$\begin{aligned} \left\| a^i - e_{n+3} \right\| &\leqslant \left\| a^i - a^i_{(m,k)} \right\| + \left\| a^i_{(m,k)} - e_{n+3} \right\| \\ &< \epsilon + \frac{\alpha_{(m,k)} + 1}{2} < \epsilon + \frac{\alpha + \epsilon + 1}{2} = \frac{\alpha + 3\epsilon + 1}{2} < 1. \end{aligned}$$

Remark 2. Of course, since Amir's characterization (15.14) is false, parts (1) of Theorems 4.1 and 4.2 of [3], which are extensions of it, are false, too.

2. On Amir's characterization (15.15)

One of the most celebrated characterizations of inner product spaces is Kakutani–Blaschke's [1, 12.4]: *A normed linear space X of dimension at least three is an inner product space if and only if for every two-dimensional subspace Y of X there exists a norm-one projection of X onto Y.* In [4, Proposition 4], Klee deepens in the meaning of this characterization. The following lemma is essentially a reformulation of Klee's result. Above all technicalities, it is important to understand the geometrical meaning.

Lemma 4. *Let $(X, \|\cdot\|)$ be a three-dimensional normed linear space, let x^* be a nonnull linear form on X, and let us denote by E the two-dimensional subspace $\{x \in X\colon x^*(x)$*

$= 0\}$. *Assume that there is no norm-one projection from X onto E. Then there is a projection P from X onto E and there are three unit vectors u_1, u_2, u_3, in $E^+ = \{x \in X\colon\ x^*(x) \geqslant 0\}$ in such a way that*

$$\|P(u_i)\| > 1$$

for $i = 1, 2, 3$, and that $\mathbf{0}$ *is interior* (*relative to E*) *to the triangle whose vertices are $P(u_1)$, $P(u_2)$, $P(u_3)$.*

Proof. Let us denote by B the unit ball of X, and by B^+ the set $B \cap E^+$. By [4, Proposition 4], there exits a projection P from X onto E such that $\mathbf{0}$ is interior (relative to E) to the convex hull of $P(B^+)\backslash B^+$. By Carathéodory lemma [5, Theorem 17.1], we deduce that there are three vectors v_1, v_2, v_3, in B^+ such that

$$\|P(v_i)\| > 1$$

for $i = 1, 2, 3$, and that $\mathbf{0}$ is a convex combination of $P(v_1)$, $P(v_2)$, $P(v_3)$. Last inequality implies that all the v_i's are nonzero, so we can take $u_i = v_i/\|v_i\|$. Of course the u_i's are norm one vectors in $B^+ \subset E^+$. Besides,

$$\|P(u_i)\| = \left\| P\left(\frac{v_i}{\|v_i\|}\right)\right\| = \frac{\|P(v_i)\|}{\|v_i\|} > \frac{1}{\|v_i\|} \geqslant 1$$

for $i = 1, 2, 3$. Moreover, since $\mathbf{0}$ is a convex combination of $P(v_1)$, $P(v_2)$, $P(v_3)$, it is also a convex combination of $P(u_1) = (1/\|v_1\|)P(v_1)$, $P(u_2) = (1/\|v_2\|)P(v_2)$, $P(u_3) = (1/\|v_3\|)P(v_3)$. To complete our proof we only have to show that $\mathbf{0}$ is *not* a convex combination of *two* of the $P(u_i)$'s, since this guarantees that $\mathbf{0}$ is in the *interior* (*relative to E*) of the triangle whose vertices are $P(u_1)$, $P(u_2)$, $P(u_3)$. If we assume, for instance, that $\mathbf{0}$ is a convex combination of $P(u_1)$ and $P(u_2)$, then there exists $\lambda \in (0, 1]$ such that either

$$P(u_1) = -\lambda P(u_2) \quad \text{or} \quad P(u_2) = -\lambda P(u_1).$$

Let us suppose we are in the first situation. Let us denote $x = P(u_1) = P(-\lambda u_2)$. Take $w \in \operatorname{Ker}(P) \cap E^+$, $w \neq 0$ (for example, take $w = u_1 - P(u_1)$). Since u_1 belongs to E^+ and $-\lambda u_2$ belongs to $E^- = \{x \in X\colon\ x^*(x) \leqslant 0\}$, there are $\alpha, \beta > 0$ such that $u_1 = x + \alpha w$ and $-\lambda u_2 = x - \beta w$. Therefore, x belongs to $[u_1, -\lambda u_2]$. But u_1 and $-\lambda u_2$ are in the unit ball and so we deduce that x is in the unit ball, too. This is a contradiction because $\|x\| = \|P(u_1)\| > 1$. □

Now we can give a proof of Amir's equivalence.

Theorem 1. *Let X be a normed linear space of dimension at least three. Then X is an IPS if and only if X satisfies* (15.15)*.*

Proof. Of course, if X is IPS then it satisfies (15.15), so we only have to prove the converse.

Suppose X is not an IPS. We may assume it has dimension three. By Kakutani–Blaschke criterion [1, 12.4], it has a two-dimensional subspace E such that X admits no norm-one projection onto E. Take a linear form x^* on X such that $E = \operatorname{Ker}(x^*)$. By the preceding lemma, there is a projection P from X onto E and there are three unit vectors u_1, u_2, u_3 in $E^+ = \{x \in X\colon x^*(x) \geqslant 0\}$ such that

$$\|P(u_i)\| > 1$$

for $i = 1, 2, 3$, and that $\mathbf{0}$ is interior (relative to E) to the triangle whose vertices are $P(u_1)$, $P(u_2)$, $P(u_3)$. Take

$$a_i = \frac{P(u_i)}{\|P(u_i)\|}$$

for $i = 1, 2, 3$. It is clear that $\mathbf{0}$ belongs to the convex hull of $\{a_1, a_2, a_3\}$.

To finish our proof we only have to show that $\mathbf{0}$ in *not* Chebyshev center of $\{a_1, a_2, a_3\}$. Take a nonzero vector a in $E^+ \cap \operatorname{Ker}(P)$. There are *positive* numbers β_1, β_2, β_3 such that $u_i = P(u_i) + \beta_i a$ for each $i = 1, 2, 3$. Hence, we have $\|u_i\| = \|P(u_i) + \beta_i a\| = 1 < \|P(u_i)\|$. Therefore,

$$\left\| \frac{P(u_i)}{\|P(u_i)\|} + \frac{\beta_i}{\|P(u_i)\|} a \right\| = \|a_i + t_i a\| < 1$$

for $i = 1, 2, 3$, where $t_i = \beta_i / \|P(u_i)\|$. If $t \in]0, t_i]$, the convexity of the map $t \mapsto \|a_i + ta\|$ guarantees that $\|a_i + ta\| < 1$. Let us denote

$$\theta = \min\{t_i\colon i = 1, 2, 3\}.$$

We have $\|a_i + \theta a\| < 1$ for each $i = 1, 2, 3$. Therefore, $\mathbf{0}$ cannot be Chebyshev center of $\{a_1, a_2, a_3\}$. $\square$

3. On hull properties

As we have already mentioned, this section deals with optimal location properties introduced by Durier in [3]. Let us begin with some notations.

Given a subset A of X, we denote by $\operatorname{aff}(A)$ (respectively, $\operatorname{conv}(A)$) the affine (respectively, convex) hull of A,

We will say that a norm γ on $\mathbb{R}^n$ is *monotone* if for each $u = (u_i)_{1 \leqslant i \leqslant n}$ and $v = (v_i)_{1 \leqslant i \leqslant n}$ in $\mathbb{R}^n$ such that $0 \leqslant u_i \leqslant v_i$ for $1 \leqslant i \leqslant n$ one has $\gamma(u) \leqslant \gamma(v)$. For instance, classical ℓ_p-norms are monotone for $1 \leqslant p \leqslant \infty$. A monotone norm on $\mathbb{R}^n$ will be called a *monotone n-norm*.

Given a monotone n-norm γ, an n-family $A = (a_1, \ldots, a_n)$ of points in X and an n-family $\omega = (\omega_1, \ldots, \omega_n)$ of positive numbers, we consider the *objective function* $G_\omega^\gamma(A)$ defined on X by

$$G_\omega^\gamma(A)(x) = \gamma\bigl(\omega_1\|x - a_1\|, \ldots, \omega_n\|x - a_n\|\bigr)$$

for each $x \in X$. The set of points (possibly empty) where $G_\omega^\gamma(A)$ attains its minimum is denoted by $M_\omega^\gamma(A)$. This is the *set of optimal locations*.

Notice that if we take $\gamma(u) = \|u\|_\infty = \sup_{1 \leqslant i \leqslant n} |u_i|$ and $\omega = (1, 1, \ldots, 1)$, then $M_\omega^\gamma(A)$ (which is usually denoted $M_1^{\ell_\infty}(A)$) is the set of all Chebyshev centers of A, and if x_0 is Chebyshev center of A then $G_\omega^\gamma(A)(x_0)$ is just the Chebyshev radius of A.

We have the following definition:

Definition 1 (Durier [3, Definition 3.1]). Let γ be a monotone n-norm. We say that X satisfies $(\mathrm{CvHP})_n^\gamma$, called *a convex hull property* (respectively, $(\mathrm{AfHP})_n^\gamma$, called an *affine hull property*) if, for every n-family A in X and for every positive n-family ω, we have

$$M_\omega^\gamma(A) \cap \mathrm{conv}(A) \neq \emptyset \quad \big(\text{respectively,} \quad M_\omega^\gamma(A) \cap \mathrm{aff}(A) \neq \emptyset\big).$$

It is immediate that if X satisfies $(\mathrm{CvHP})_n^\gamma$, then it satisfies $(\mathrm{AfHP})_n^\gamma$, too. Then, it is natural to ask whether the converse is true or not. We show now that the answer is affirmative. They are actually equivalent properties. This will follow from the following proposition, which is a well known consequence of Helly's theorem (see, for instance, Proposition 3.2 of [3]). We include proof for the sake of completeness.

Proposition 1. *Let $(X, \|\cdot\|)$ be a two-dimensional normed linear space, and let $x_1, \ldots, x_n$ be n points in X. Then for each $y \in X$ there is a point x_0 in the convex hull of $x_1, \ldots, x_n$, such that*

$$\|x_i - x_0\| \leqslant \|x_i - y\|$$

for $i = 1, \ldots, n$.

Proof. The result is trivial for $n \leqslant 2$, so we will assume $n \geqslant 3$. Let us denote by T the convex hull of $x_1, \ldots, x_n$. For $i = 1, \ldots, n$ take $r_i = \|x_i - y\|$, and denote $B_i = \{x \in X\colon \|x_i - x\| \leqslant r_i\}$. It is easy to check that each three members of the family of convex sets $\{B_1, \ldots, B_n, T\}$ have nonempty intersection. Then, Helly's theorem [5, Theorem 21.6] in dimension two guarantees that the family has nonempty intersection, that is, there exists $x_0 \in T \cap B_1 \cap \cdots \cap B_n$. It is clear that x_0 satisfies the desired property. □

Now we can easily get the following results.

Proposition 2. *Properties $(\mathrm{AfHP})_3^\gamma$ and $(\mathrm{CvHP})_3^\gamma$ are equivalent.*

Proof. Of course, $(\mathrm{CvHP})_3^\gamma$ implies $(\mathrm{AfHP})_3^\gamma$, so we only have to prove the converse. Assume that a normed linear space X enjoys $(\mathrm{AfHP})_3^\gamma$. Let $A = \{a_1, a_2, a_3\}$ be a subset of X and let ω be a 3-family of positive real numbers. Taking a translation if necessary, we can assume that the affine space generated by $A = \{a_1, a_2, a_3\}$ passes through $\mathbf{0}$, that is, it is a (two-dimensional) vector space. Let us denote by F this normed linear space. By our hypothesis, there exists $x_0 \in F$ such that $G_\omega^\gamma(A)(x_0) \leqslant G_\omega^\gamma(A)(x)$ for each $x \in X$. By the preceding proposition there exists $x_0' \in \mathrm{conv}(A)$ such that $\|x_0' - a_i\| \leqslant \|x_0 - a_i\|$ for $i = 1, 2, 3$. Since $\omega_i > 0$ for each $i = 1, 2, 3$ and γ is monotone, we have $G_\omega^\gamma(A)(x_0') \leqslant G_\omega^\gamma(A)(x_0) \leqslant G_\omega^\gamma(A)(x)$ for each $x \in X$. This concludes our proof. □

Proposition 3. *Let X be a normed linear space of dimension at least three and let γ be monotone 3-norm. Then the following are equivalent*:

(i) *X satisfies* $(\mathrm{AfHP})_3^\gamma$.
(ii) *X satisfies* $(\mathrm{CvHP})_3^\gamma$.
(iii) *X is an IPS.*

Proof. (i) $\Rightarrow$ (ii) follows from the preceding proposition.
(ii) $\Rightarrow$ (iii) follows from Theorem 3.8 of [3].
(iii) $\Rightarrow$ (i) is well known; see, for instance, Proposition 3.2 of [3]. $\square$

Theorem 2. *Let X be a normed linear space of dimension at least three and let γ be monotone n-norm, where $n \geqslant 3$. Then the following are equivalent*:

(i) *X satisfies* $(\mathrm{AfHP})_n^\gamma$.
(ii) *X satisfies* $(\mathrm{CvHP})_n^\gamma$.
(iii) *X is an IPS.*

Proof. Assume that (i) (respectively, (ii)) is satisfied. Then, by Proposition 3.3 of [3] X satisfies $(\mathrm{AfHP})_3^{\gamma'}$ (respectively, $(\mathrm{CvHP})_3^{\gamma'}$) for some monotone 3-norm γ'. In any case, we deduce from the preceding proposition that X is an IPS.

Converses are very well known. See, for instance, Proposition 3.2 of [3]. $\square$

Remark 3. The preceding theorem gives an extension of Theorem 3.5 of [3].

Let us summarize the situation of $(\mathrm{AfHP})_n^\gamma$ and $(\mathrm{CvHP})_n^\gamma$.

Corollary 1. *Let X be a normed linear space, let n be a natural number and let γ be a monotone n-norm. Then we have*:

(1) *For $n = 1$ and $n = 2$ all linear normed spaces X enjoy* $(\mathrm{AfHP})_n^\gamma$ *and* $(\mathrm{CvHP})_n^\gamma$.
(2) *For $n \geqslant 3$ all two-dimensional linear normed spaces X enjoy* $(\mathrm{AfHP})_n^\gamma$ *and* $(\mathrm{CvHP})_n^\gamma$.
(3) *For $n \geqslant 3$, if the dimension of X is at least three, then X enjoys either* $(\mathrm{AfHP})_n^\gamma$ *and* $(\mathrm{CvHP})_n^\gamma$ *if and only if X is an IPS.*

Proof. The proof of the first assertion is straightforward, the second one is Proposition 3.2 of [3] (which follows from Proposition 1 stated above) and the third one is the preceding theorem. $\square$

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